Power absorption during forced oscillations

- Forced oscillation of undamped oscillator: No net power absorption (per osc. cycle) as system does not dissipate energy.

- Forced oscillation of real oscillator w/ damping in steady state:

> The power absorbed by the oscillator to sustain its motion is exactly equal to the rate that the energy is dissipated.

- We assume $F_{\text{damping}} \sim$ velocity, so we start by examining behavior of velocity
- Solution to damped, FHO: $x = A(\omega)\cos(\omega t - \delta)$ [see last lecture]

$$A(\omega) = \frac{a\omega_0^2}{\left[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2\right]^{1/2}}$$

$$\rightarrow v = \frac{dx}{dt}$$

$$= -A(\omega)\omega\sin(\omega t - \delta)$$

$$= -v_0(\omega)\sin(\omega t - \delta)$$, where we define $v_0(\omega) \equiv A(\omega)\omega$

- We can think of $v_0(\omega)$ as the "amplitude" of velocity oscillations, just like $A(\omega)$ is amplitude of displacement oscillations.

$$\rightarrow v_0(\omega) = \frac{a\omega_0^2}{\left[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2\right]^{1/2}} = \frac{a\omega_0}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2\omega_0^2 + \gamma^2\right]^{1/2}}$$

~~[w/ some algebra]~~

w/ some algebra

- Energy of an oscillator is dissipated b/c it does work against the damping force at rate (damping force × velocity)

$$\rightarrow \frac{dE}{dt} = F_d v \quad , \quad \text{where } F_d = -bv$$

$$= -b[v(t)]^2 = \underline{\text{Power dissipated}}$$

↑ b/c velocity is time-dependent

- We want power absorbed, which equals −(power dissipated)

$$\rightarrow P(t) = b[v_0(\omega)]^2 \sin^2(\omega t - \delta)$$

↑ instantaneous power absorbed

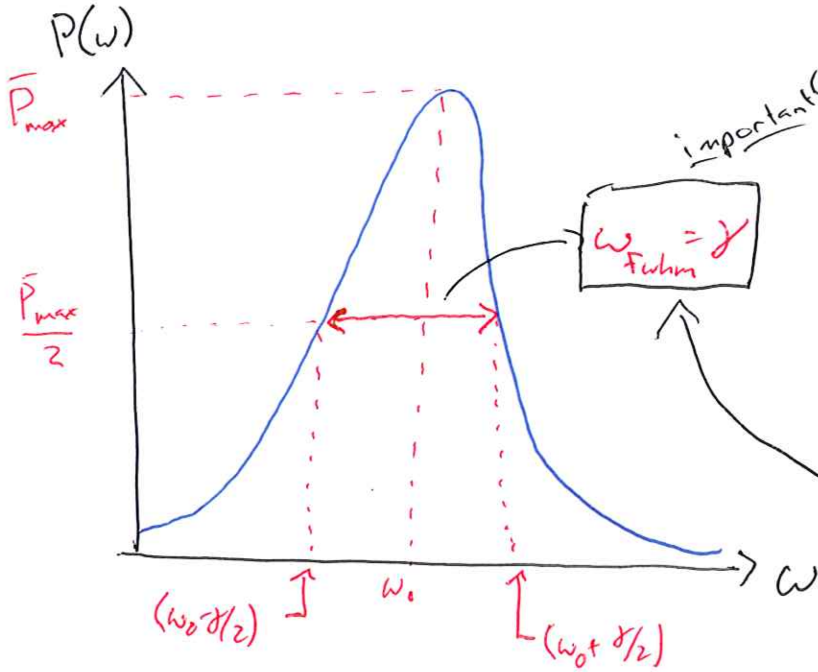
- Since $P(t)$ varies in time, more appropriate to consider ~~power~~ average power absorbed over one cycle of oscillation b/t times t_0 and $t_0 + T$, where T is period:

$$\overline{P(\omega)} = \frac{1}{T}\int_{t_0}^{t_0+T} P(t)\,dt$$

$$\Rightarrow \overline{P(\omega)} = \frac{b[v_0(\omega)]^2}{T}\underbrace{\int_{t_0}^{t_0+T}\sin^2(\omega t - \delta)\,dt}_{=\frac{T}{2}} = \frac{b\,v_0(\omega)^2}{2}$$

· Plugging in $v_0(\omega)$ and using $b = m\gamma$, $\omega_0^2 = k/m$, $a = F_0/k$

$$\rightarrow \overline{P(\omega)} = \frac{\omega^2 F_0^2 \gamma}{2m\left[(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2\right]}$$



important!

$$\boxed{\omega_{fwhm} = \gamma}$$

"Full width half max"

See that:

$$\bar{P}(\omega) \xrightarrow{\omega \to 0} 0$$

$$\bar{P}(\omega) \xrightarrow{\omega \to \infty} 0$$

$$\bar{P}(\omega_0) = \bar{P}_{max}$$

· For frequencies close to ω_0 can approximate: $[\omega \sim \omega_0]$

$$(\omega^2 - \omega_0^2) = \overbrace{(\omega_0 + \omega)}^{\sim 2\omega_0}(\omega_0 - \omega) \approx 2\omega_0(-\Delta\omega) \quad ; \quad \Delta\omega \doteq \omega - \omega_0$$

$$\rightarrow \boxed{\bar{P}(\omega) \simeq \frac{F_0^2}{2m\gamma(4\Delta\omega^2/\gamma^2 + 1)}}$$

power resonance curve

$$\bar{P}_{max} = \frac{F_0^2}{2m\gamma}$$ occurs when $\Delta\omega = 0$ (ie, on resonance) $\omega = \omega_0$

$\rightarrow \cancel{\frac{\bar{P}_{max}}{2} \neq 0}$ $\quad \underbrace{\omega_{fwhm}}_{} = 2\Delta\omega = \gamma = \omega_0/Q \longrightarrow$

can arrive at this by considering when get $\frac{\bar{P}_{max}}{2}$

Q-factor

$\frac{\bar{P}_{max}}{2}$ occur when $\frac{4\Delta\omega^2}{\gamma^2} = 1 \Rightarrow 2\Delta\omega = \gamma$

$\rightarrow \omega_{FWHM} = 2\Delta\omega = \gamma = \omega_0 / Q$

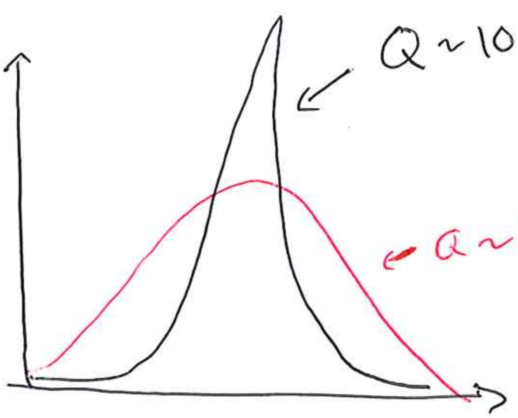
Q-factor

· Can then write Q-factor as:

$$\Rightarrow Q = \frac{\omega_0}{\gamma} = \frac{\omega_0}{\omega_{FWHM}} = \frac{\text{resonance frequency}}{\text{Full width at half max of power curve}}$$

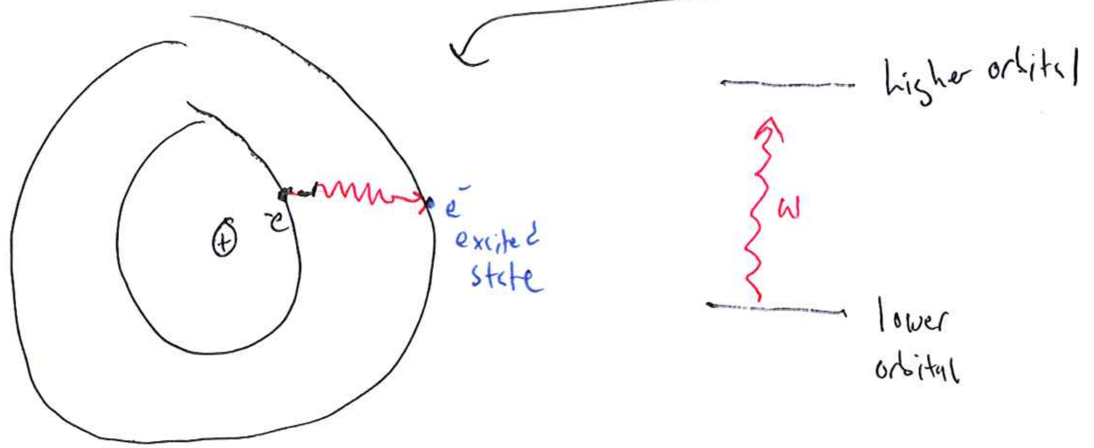
· Can rewrite power w/ Q:

$$\bar{P}(\omega) = \frac{F_0^2}{2m\omega_0 Q\left[4(\Delta\omega/\omega_0)^2 + 1/Q^2\right]}$$



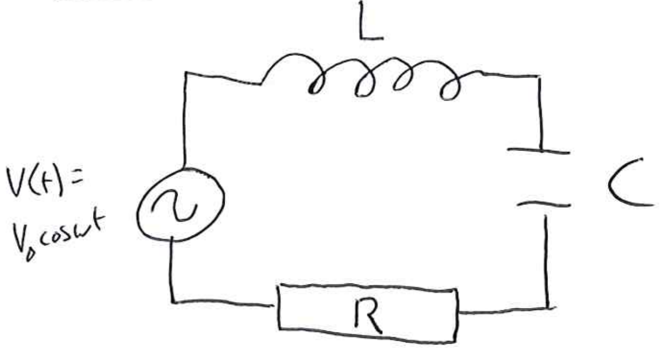
Power resonance curves common in physics:

- mechanical osc.
- electrical osc.
- atomic physics



An atom is a high-Q oscillator!

Resonance in electrical circuits



Eq. of motion:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V_0 \cos\omega t$$

- Completely analogous to mech. case: $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0\cos\omega t$

w/ $m \to L$, $b \to R$, $k \to \frac{1}{C}$

$$\Rightarrow \omega_0^2 = \frac{1}{LC}, \quad \gamma = \frac{R}{L}, \quad Q = \frac{\omega_0}{\gamma} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

- Solution is $q = q_0(\omega)\cos(\omega t - \delta)$

$$\text{w/ } q_0(\omega) = \frac{V_0}{\omega\left[\left(1/\omega C - \omega L\right)^2 + R^2\right]^{1/2}}$$

- Via $V_c = q/C$, voltage across capacitor is

$$V_c(t) = V_{c,0}(\omega)\cos(\omega t - \delta) \qquad \text{w/ } V_{c,0}(\omega) = \frac{q_0(\omega)}{C}$$

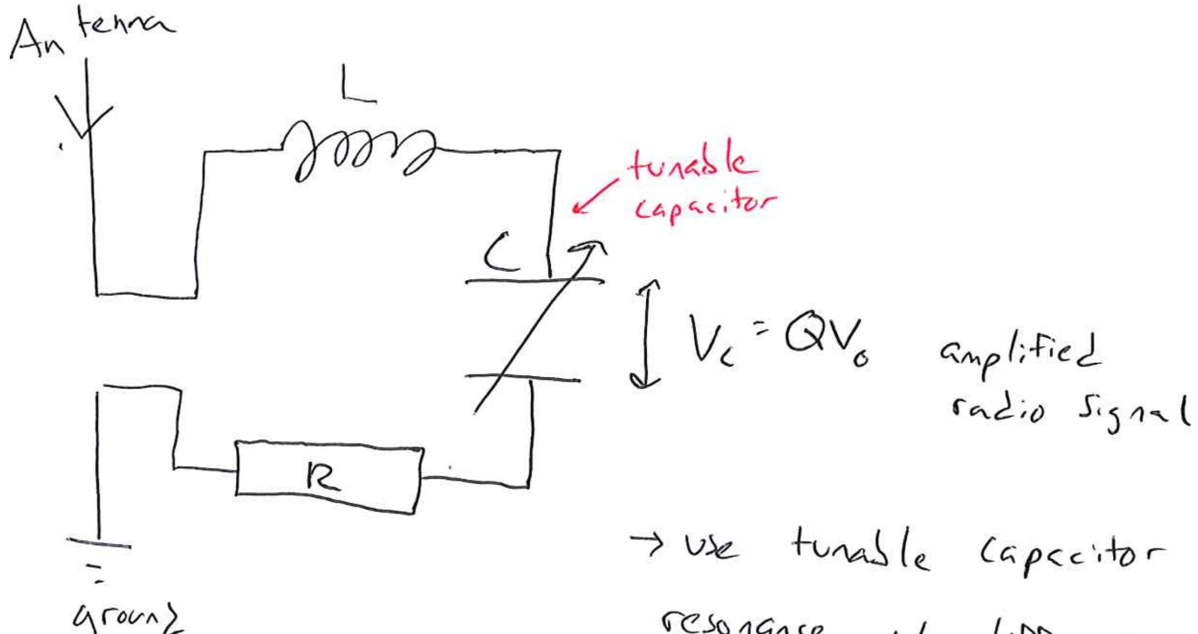
- At resonance $\omega = \omega_0$

$$V_{c,0}(\omega) = \frac{V_0}{R\omega_0 C} = Q V_0$$

$\to$ Drive amplitude V_0 gets amplified by factor of Q!

· Typical Q-value in electrical circuit ~ 200

· Example: radio ~~receiuer~~ receiver:



→ use tunable capacitor to ~~amplify~~ tune into resonance w/ different radio frequencies

Complex representation of oscillators

Complex numbers: a whirlwind review

→ Please read King Ch. 3.6! This goes over complex numbers in more detail.

$$\underset{\substack{\uparrow \\ \text{complex} \\ \#}}{z} = \underset{\substack{\uparrow \\ \text{"real"} \\ \text{part}}}{x} + i\underset{\substack{\uparrow \\ \text{"imaginary} \\ \text{part"}}}{y} \qquad \text{where } i = \sqrt{-1}$$

$$x = \mathrm{Re}(z), \quad y = \mathrm{Im}(z)$$

- Useful quantity: "complex conjugate"
  - → obtained by replacing i w/ $-i$

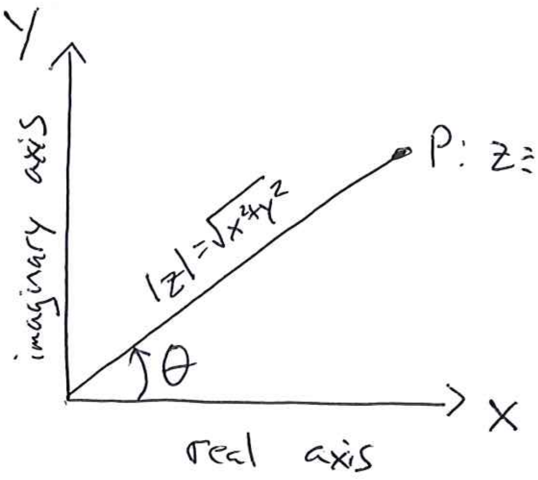
$$z^* = x - iy$$

- Also useful: $zz^* = x^2 + y^2 = |z|^2$

  ↳ $|z| = \sqrt{zz^*}$ "modulus of z"

→ Those are basic ~~[illegible]~~ rules for working w/ complex #s

· The meaning is brought out via geometrical interpretation



P: $z = (r, \theta)$ (polar coordinates)

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} \; ; \; \sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$

$\rightarrow$ $x = r\cos\theta$ $\quad$ $y = r\sin\theta$

$r = \sqrt{x^2+y^2}$

Important: Euler eq.

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

[can prove this via power series expansion]

$\rightarrow$ $z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$\hookrightarrow$ Any complex # can be represented as a magnitude (r) and phase (θ) in the complex plane using polar coordinates

Using complex numbers to solve differential eqs.

[see King 3.6.2 for more intro on this]

- Consider the eq. for a damped, forced HO:

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

→ Previously we solved this assuming solution

$x = A(\omega)\cos(\omega t - \delta)$. There was lots of algebra...

- Much easier to use complex #s. Replace above diff. eq. w/ complex version:

$$\frac{d^2z}{dt^2} + \gamma \frac{dz}{dt} + \omega_0^2 z = \frac{F_0}{m} e^{i\omega t}$$

- We do this knowing that, at the end of the day, we only care about the real part of the solution.

- Assume solution has the form $z = A(\omega) e^{i(\omega t - \delta)}$

- Plug this in to above diff. eq. $\frac{dz}{dt} = i\omega A(\omega) e^{i(\omega t - \delta)}$

$$\frac{d^2z}{dt^2} = i^2\omega^2 A(\omega) e^{i(\omega t - \delta)}$$
$$= -\omega^2 A(\omega) e^{i(\omega t - \delta)}$$

→

$$\rightarrow \left[-\omega^2 A(\omega) + i\gamma\omega A(\omega) + \omega_0^2 A(\omega)\right] e^{i(\omega t - \delta)} = \frac{F_0}{m} e^{i\omega t}$$

- Divide through by $e^{i(\omega t - \delta)}$

$$\rightarrow (\omega_0^2 - \omega^2) A(\omega) + i\gamma\omega A(\omega) = \frac{F_0}{m} e^{i\delta}$$

- Next, we separate this eq. into the real & imaginary parts

$$\Rightarrow (1) \quad (\omega_0^2 - \omega^2) A(\omega) = \frac{F_0}{m} \cos\delta \qquad \text{(real)}$$

$$(2) \quad \gamma\omega A(\omega) = \frac{F_0}{m} \sin\delta$$

→ Both of these eqs. are individually true, so the ratio is also true

$$\frac{(2)}{(1)} = \tan\delta = \frac{\gamma\omega}{(\omega_0^2 - \omega^2)}$$

and $$A(\omega) = \frac{F_0/m}{\left[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2\right]^{1/2}}$$

- Same result as before, but much easier to get there!